

SAMELSON PRODUCTS IN p -REGULAR $\mathrm{SO}(2n)$ AND ITS HOMOTOPY NORMALITY

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ABSTRACT. A Lie group is called p -regular if it has the p -local homotopy type of a product of spheres. (Non)triviality of the Samelson products of the inclusions of the factor spheres into p -regular $\mathrm{SO}(2n)_{(p)}$ is determined, which completes the list of (non)triviality of such Samelson products in p -regular simple Lie groups. As an application, we determine the homotopy normality of the inclusion $\mathrm{SO}(2n-1) \rightarrow \mathrm{SO}(2n)$ in the sense of James at any prime p .

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let G be a compact connected Lie group. By the classical result of Hopf, it is well known that there is a rational homotopy equivalence

$$G \simeq_{(0)} S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$$

where $n_1 \leq \cdots \leq n_\ell$. The sequence $n_1 \leq \cdots \leq n_\ell$ is called the type of G . Here is the list of the types of simple Lie groups.

$\mathrm{SU}(n)$	$2, 3, \dots, n$	G_2	$2, 6$
$\mathrm{SO}(2n+1)$	$2, 4, \dots, 2n$	F_4	$2, 6, 8, 12$
$\mathrm{Sp}(n)$	$2, 4, \dots, 2n$	E_6	$2, 5, 6, 8, 9, 12$
$\mathrm{SO}(2n)$	$2, 4, \dots, 2n-2, n$	E_7	$2, 6, 8, 10, 12, 14, 18$
		E_8	$2, 8, 12, 14, 18, 20, 24, 30$

Serre generalizes the above rational homotopy equivalence to a p -local homotopy equivalence such that when G is semisimple and $G_{(p)}$ is simply connected, there is a p -local homotopy equivalence

$$(1.1) \quad G \simeq_{(p)} S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$$

if and only if $p \geq n_\ell$, in which case G is called p -regular. In this paper we are interested in the standard multiplicative structure of the p -localization $G_{(p)}$ when G is p -regular, and then we assume that G is a simple Lie group in the above table and is p -regular throughout this section.

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Recall that for a homotopy associative H-space X with inverse and maps $\alpha: A \rightarrow X, \beta: B \rightarrow X$, the correspondence

$$A \wedge B \rightarrow X, \quad (x, y) \mapsto \alpha(x)\beta(y)\alpha(x)^{-1}\beta(y)^{-1}$$

is called the Samelson product of α, β in X and is denoted by $\langle \alpha, \beta \rangle$. One easily sees that in investigating the multiplicative structure of $G_{(p)}$, the Samelson products $\langle \epsilon_i, \epsilon_j \rangle$ play the fundamental role as in [KK], where ϵ_i is the inclusion $S^{2n_i-1} \rightarrow S_{(p)}^{2n_1-1} \times \cdots \times S_{(p)}^{2n_\ell-1} \simeq G_{(p)}$ into the i -th factor. So it is our task to determine (non)triviality of these Samelson products. In this direction, Bott [B] studied the order of a certain class of Samelson products in $SU(n)$ and $Sp(n)$, for example.

We here make a remark on the choice of ϵ_i which depends on the p -local homotopy equivalence (1.1). Recall from [T, Theorem 13.4] that

$$(1.2) \quad \pi_*(S_{(p)}^{2m-1}) = 0 \quad \text{for} \quad 2m-1 < * < 2m+2p-4.$$

Then we see that $\pi_{2n_i-1}(G_{(p)})$ is a free $\mathbb{Z}_{(p)}$ -module for all i , and so $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for all i and $G \neq SO(2n)$ since the entries of the type are distinct for $G \neq SO(2n)$ as in the above table. Hence for $G \neq SO(2n)$ we may choose any generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ as ϵ_i . For $G = SO(2n)$ we will make an explicit choice of ϵ_i below.

We first consider the Samelson products $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ when G is the classical group except for $SO(2n)$.

Theorem 1.1. *Let G be the p -regular classical group except for $SO(2n)$, and let ϵ_i be a generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for the type $\{n_1, \dots, n_\ell\}$ of G . Then*

$$\langle \epsilon_i, \epsilon_j \rangle \neq 0 \quad \text{if and only if} \quad n_i + n_j > p.$$

Proof. If $G = SU(n), Sp(n)$, nontriviality of the Samelson products follows from the result of Bott [B] and triviality follows from the fact that $\pi_{2*}(G_{(p)}) = 0$ for $* < p$ which is deduced from (1.2). Since there is a homotopy equivalence as loop spaces $Sp(n)_{(p)} \simeq SO(2n+1)_{(p)}$ due to Friedlander [F], the case of $SO(2n+1)_{(p)}$ is the same as $Sp(n)_{(p)}$. \square

We next consider the Samelson products $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ when G is the exceptional Lie group. Some of these Samelson products are calculated in [HK2, KK], and (non)triviality of all these Samelson products is determined in [HKO] as follows.

Theorem 1.2 ([HKO]). *Let G be a p -regular compact connected exceptional simple Lie group, and let ϵ_i be a generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for the type $\{n_1, \dots, n_\ell\}$ of G . Then*

$$\langle \epsilon_i, \epsilon_j \rangle \neq 0 \quad \text{if and only if} \quad n_i + n_j = n_k + p - 1 \text{ for some } k.$$

Thus the only remaining case is $SO(2n)$. The purpose of this paper is to show that a sufficient condition for nontriviality of the Samelson products $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ (Lemma 2.1) used in [KO, HK1, HK2, HKO] is actually a necessary and sufficient condition, and is to apply it to determination of (non)triviality of all the Samelson products $\langle \epsilon_i, \epsilon_j \rangle$ in $SO(2n)_{(p)}$. The difficulty

of this case is caused by the middle dimensional sphere $S_{(p)}^{2n-1}$ in $\mathrm{SO}(2n)_{(p)}$ which vanishes by the inclusion $\mathrm{SO}(2n) \rightarrow \mathrm{SO}(2n+1)$. we choose the maps ϵ_i . Let ϵ_i be the composite

$$S^{4i-1} \rightarrow \mathrm{SO}(2n-1)_{(p)} \xrightarrow{\mathrm{incl}} \mathrm{SO}(2n)_{(p)}$$

for $i = 1, \dots, n-1$, where the first arrow is a generator of $\pi_{4i-1}(\mathrm{SO}(2n-1)_{(p)}) \cong \mathbb{Z}_{(p)}$. Let $\theta: S^{2n-1} \rightarrow \mathrm{SO}(2n)_{(p)}$ be the map corresponding to the adjoint of the fiber inclusion of the canonical homotopy fiber sequence

$$S^{2n} \rightarrow \mathrm{BSO}(2n) \rightarrow \mathrm{BSO}(2n+1).$$

There are only two results on Samelson products in $\mathrm{SO}(2n)$ involving θ : Mahowald [Ma] showed that the Samelson product $\langle \theta, \theta \rangle \in \pi_{4n-2}(\mathrm{SO}(2n))$ has order $(2n-1)!/8$ or $(2n-1)!/4$ according as n is even or odd. Hamanaka and Kono [HK1] showed that the Samelson product $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle \in \pi_{4n-2}(\mathrm{SO}(2n)_{(p)})$ is non-trivial when $p \leq 2n-1$. Our main result determines (non)triviality of all Samelson products of ϵ_i and θ in p -regular $\mathrm{SO}(2n)$.

Theorem 1.3. *Let ϵ_i, θ be the above maps into $\mathrm{SO}(2n)_{(p)}$ for p -regular $\mathrm{SO}(2n)$. All nontrivial Samelson products of ϵ_i, θ in $\mathrm{SO}(2n)_{(p)}$ are*

$$\langle \epsilon_i, \epsilon_j \rangle \quad \text{for } 2i + 2j > p \quad \text{and} \quad \langle \epsilon_{n-1}, \theta \rangle = \langle \theta, \epsilon_{n-1} \rangle, \quad \langle \theta, \theta \rangle \quad \text{for } p = 2n-1.$$

Recall that an H-map $f: X \rightarrow Y$ between homotopy associative H-spaces with inverse X, Y is homotopy normal in the sense of James [J] if the Samelson product $\langle f, 1_Y \rangle$ compressed to X through f up to homotopy. This is a generalization of the inclusion of a normal subgroup. James proved that $\mathrm{O}(n)$ is not homotopy normal in $\mathrm{O}(n+1)$ when $n \geq 2$ using the mod 2 cohomology. His proof implies that the 2-localization $\mathrm{SO}(n)_{(2)}$ is not homotopy normal in $\mathrm{SO}(n+1)_{(2)}$ when $n \geq 2$. As an application of Theorem 1.3 we will prove:

Theorem 1.4. *The inclusion $\iota_{(p)}: \mathrm{SO}(2n-1)_{(p)} \rightarrow \mathrm{SO}(2n)_{(p)}$ is homotopy normal if and only if $p > 2n-1$.*

For $p > 2n-1$, we can prove the following stronger result.

Theorem 1.5. *For $p > 2n-1$, the map $\iota_{(p)} \cdot \theta: \mathrm{SO}(2n-1)_{(p)} \times S_{(p)}^{2n-1} \rightarrow \mathrm{SO}(2n)_{(p)}$ is an H-equivalence, where $S_{(p)}^{2n-1}$ is a homotopy associative and homotopy commutative H-space.*

2. DETECTING SAMELSON PRODUCTS BY THE STEENROD OPERATIONS

Let G be a p -torsion free connected finite loop space of type $n_1 \leq \dots \leq n_\ell$ throughout this section where the type of a finite loop space is similarly defined. We set notation for G . Since G is p -torsion free, we have

$$H^*(BG_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[x_1, \dots, x_\ell], \quad |x_i| = 2n_i.$$

We fix this presentation of the mod p cohomology of $BG_{(p)}$. Note that

$$H^*(G_{(p)}; \mathbb{Z}/p) = \Lambda(e_1, \dots, e_\ell)$$

for the suspension e_i of x_i . For each i , we take $\epsilon_i \in \pi_{2n_i-1}(G_{(p)})$ which is not divisible by non-units in $\mathbb{Z}_{(p)}$ such that

$$(2.1) \quad (\Sigma \epsilon_i)^* \circ \iota_1^*(x_j) = \begin{cases} h_i \Sigma u_{2n_i-1} & i = j \\ 0 & i \neq j \end{cases}$$

for some $h_i \in \mathbb{Z}_{(p)}$, where $\iota_1: \Sigma G_{(p)} \rightarrow BG_{(p)}$ is the canonical map and u_k is a generator of $H^k(S^k; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$. The following lemma is first used in [KO] and is the main tool in the proof of Theorem 1.2 given in [HKO]. Here we reproduce the proof for completeness of the present paper.

Lemma 2.1 ([KO, Proof of Theorem 1.1]). *Suppose that h_i and h_j are units in $\mathbb{Z}_{(p)}$. If $\mathcal{P}^1 x_k$ is decomposable and includes the term $cx_i x_j$ ($c \neq 0$), the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is nontrivial.*

Proof. Suppose $\langle \epsilon_i, \epsilon_j \rangle = 0$ under the assumption that $\mathcal{P}^1 x_k$ includes the term $cx_i x_j$ ($c \neq 0$). Let $\bar{\epsilon}_m: S^{2n_m} \rightarrow BG_{(p)}$ be the adjoint of ϵ_m . Then by (2.1), we have $\bar{\epsilon}_m^*(x_m) = h_m u_{2m}$. By adjointness of Samelson products and Whitehead products, the Whitehead product $[\bar{\epsilon}_i, \bar{\epsilon}_j]$ in $BG_{(p)}$ is trivial, and then there is a map $\mu: S^{2n_i} \times S^{2n_j} \rightarrow BG_{(p)}$ satisfying $\mu|_{S^{2n_i} \vee S^{2n_j}} = \bar{\epsilon}_i \vee \bar{\epsilon}_j$. So we get $\mu^*(x_i) = h_i(u_{2n_i} \otimes 1)$ and $\mu^*(x_j) = h_j(1 \otimes u_{2n_j})$, and hence

$$ch_i h_j u_{2n_i} \otimes u_{2n_j} = \mu^*(cx_i x_j) = \mu^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \mu^*(x_k) = 0$$

where the second and the last equality follows from the decomposability of $\mathcal{P}^1 x_k$ and triviality of \mathcal{P}^1 on $H^*(S^{2n_i} \times S^{2n_j}; \mathbb{Z}/p)$, respectively. This is a contradiction to $ch_i h_j \neq 0$. \square

In this lemma, the assumption on the decomposability of $\mathcal{P}^1 x_k$ cannot be removed. Here is a counterexample.

Example 2.2. We consider $SU(4)$ at the prime 3. Recall that $H^*(BSU(4); \mathbb{Z}/3) = \mathbb{Z}/3[c_2, c_3, c_4]$, where c_i denotes the i^{th} universal Chern class. By inspection, we have

$$\mathcal{P}^1 c_2 = c_2^2 + c_4.$$

For a degree reason, the inclusion $\epsilon_1: S^3 = SU(2) \rightarrow SU(4)$ satisfies $(\Sigma \epsilon_1)^* \circ \iota_1^*(c_2) = \Sigma u_3$ as in (2.1), but the Samelson product $\langle \epsilon_1, \epsilon_1 \rangle$ is trivial since $SU(2)$ commutes up to homotopy with itself in $SU(4)$.

We elaborate Lemma 2.1 to prove that its converse is true when $G_{(p)}$ is a product of spheres. The following lemma is useful to detect the nontriviality of a Samelson product when $G_{(p)}$ is decomposed into a product of a sphere and some space. The proof is independent of Lemma 2.1.

Lemma 2.3. *For integers $1 \leq i, j, k \leq \ell$, suppose that there is a map $\pi_k: G_{(p)} \rightarrow S_{(p)}^{2n_k-1}$ such that $\pi_k^*(u_{2n_k-1}) = e_k$, h_i and h_j are units in $\mathbb{Z}_{(p)}$, and $n_i + n_j = n_k + p - 1$. Then $\pi_k \circ \langle \epsilon_i, \epsilon_j \rangle \neq 0$ if and only if $\mathcal{P}^1 x_k$ includes the term $cx_i x_j$ with $c \neq 0$.*

Proof. We prove both implications simultaneously. We may suppose that $h_i = h_j = h_k = 1$. Let $P^2G_{(p)}$ be the projective plane of $G_{(p)}$, i.e. there is a cofiber sequence

$$(2.2) \quad \Sigma G_{(p)} \wedge G_{(p)} \xrightarrow{H} \Sigma G_{(p)} \xrightarrow{\rho_1} P^2G_{(p)}$$

where H is the Hopf construction. Put $\bar{x}_i = \iota_2^*(x_i)$ for the natural map $\iota_2: P^2G_{(p)} \rightarrow BG_{(p)}$. Since $\iota_1 = \iota_2 \circ \rho_1$ for the inclusion $\rho_1: \Sigma G_{(p)} \rightarrow P^2G_{(p)}$ and $\iota_1^*(x_i) = \Sigma e_i$, we have $\rho_1^*(\bar{x}_i) = \Sigma e_i$. By [L, Section 3], we also have $\delta_1^*(\Sigma^2 e_i \otimes e_j) = \bar{x}_i \bar{x}_j$ for the connecting map $\delta_1: P^2G_{(p)} \rightarrow \Sigma^2 G_{(p)} \wedge G_{(p)}$ of the cofiber sequence (2.2). Consider the map

$$\Phi = \Sigma \langle \epsilon_i, \epsilon_j \rangle - [\Sigma \epsilon_i, \Sigma \epsilon_j]: \Sigma S^{2n_i-1} \wedge S^{2n_j-1} \rightarrow \Sigma G_{(p)}$$

where $[-, -]$ denotes the Whitehead product. The map Φ is connected with the Hopf construction H through the map constructed by Morisugi [Mo, Theorem 5.1] such that there is a map $\xi: S^{2n_i-1} \wedge S^{2n_j-1} \rightarrow G_{(p)} \wedge G_{(p)}$ satisfying

$$\Phi = H \circ \Sigma \xi \quad \text{and} \quad \xi^*(e_i \otimes e_j) = u_{2n_i-1} \otimes u_{2n_j-1}.$$

Then we get a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma G_{(p)} & \xrightarrow{\rho_2} & C_\Phi & \xrightarrow{\delta_2} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1} \\ \parallel & & \downarrow \lambda_1 & & \downarrow \Sigma^2 \xi \\ \Sigma G_{(p)} & \xrightarrow{\rho_1} & P^2G_{(p)} & \xrightarrow{\delta_1} & \Sigma^2 G_{(p)} \wedge G_{(p)} \end{array}$$

whose rows are homotopy cofibrations, implying that

$$(2.3) \quad \rho_2^* \circ \lambda_1^*(\bar{x}_k) = \Sigma e_k \quad \text{and} \quad \lambda_1^*(\bar{x}_i \bar{x}_j) = \delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}).$$

We have

$$\pi_k \circ \langle \epsilon_i, \epsilon_j \rangle = c\alpha_1 \quad (c \in \mathbb{Z}/p)$$

where α_1 is a generator of $\pi_{2n_k+2p-4}(S^{2n_k-1}) \cong \mathbb{Z}/p$ [T, Proposition 13.6]. Note that $\pi_k \circ \langle \epsilon_i, \epsilon_j \rangle$ is nontrivial if and only if $c \neq 0$. Then for the map

$$\widehat{\Phi} = c\Sigma\alpha_1 - [\Sigma\pi_k \circ \epsilon_i, \Sigma\pi_k \circ \epsilon_j]: \Sigma S^{2n_i-1} \wedge S^{2n_j-1} \rightarrow \Sigma S_{(p)}^{2n_k-1}$$

there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma G_{(p)} & \xrightarrow{\rho_2} & C_\Phi & \xrightarrow{\delta_2} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1} \\ \downarrow \Sigma\pi_k & & \downarrow \lambda_2 & & \parallel \\ \Sigma S^{2n_k-1} & \xrightarrow{\rho_3} & C_{\widehat{\Phi}} & \xrightarrow{\delta_3} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1} \end{array}$$

whose rows are homotopy cofibrations. Since α_1 is detected by the Steenrod operation \mathcal{P}^1 , the mod p cohomology of $C_{\widehat{\Phi}}$ is given by

$$\widetilde{H}^*(C_{\widehat{\Phi}}; \mathbb{Z}/p) = \langle a_{2n_k}, a_{2n_i+2n_j} \rangle, \quad \mathcal{P}^1 a_{2n_k} = c a_{2n_i+2n_j}$$

such that $\delta_3^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}) = a_{2n_i+2n_j}$ and $\rho_3^*(a_{2n_k}) = \Sigma u_{2n_k-1}$. Then by (2.3), we get $\rho_2^* \circ \lambda_2^*(a_{2n_k}) = \rho_2^* \circ \lambda_1^*(\bar{x}_k)$. By the homotopy cofiber sequence $\Sigma G_{(p)} \xrightarrow{\rho_2} C_\Phi \xrightarrow{\delta_2} \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1}$ one can see that the inclusion $\rho_2: \Sigma G_{(p)} \rightarrow C_\Phi$ is injective in the mod p cohomology of dimension $2n_k$, and then we obtain $\lambda_2^*(a_{2n_k}) = \lambda_1^*(\bar{x}_k)$. By (2.3), we also have $\lambda_2^*(a_{2n_i+2n_j}) = \lambda_1^*(\bar{x}_i \bar{x}_j)$. Hence since $\mathcal{P}^1 a_{2n_k} = c a_{2n_i+2n_j}$, we get that $\mathcal{P}^1 \bar{x}_k$ includes the term $c \bar{x}_i \bar{x}_j$. Thus since $\iota_2^*(x_m) = \bar{x}_m$ for $m = 1, \dots, \ell$, $\mathcal{P}^1 x_k$ must include the term $c x_i x_j$. Therefore we have established the lemma. \square

Theorem 2.4. *Suppose $p \geq n_\ell - n_1 + 2$. Then the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ is nontrivial if and only if for some k , $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$.*

Proof. By the result of Kumpel [K], we can choose each ϵ_i such as $h_i = 1$. Then the composite

$$S^{2n_1-1} \times \dots \times S^{2n_\ell-1} \xrightarrow{\epsilon_1 \times \dots \times \epsilon_\ell} G_{(p)} \times \dots \times G_{(p)} \rightarrow G_{(p)}$$

induces a p -local homotopy equivalence where the second map is the multiplication, and we identify $G_{(p)}$ with $S_{(p)}^{2n_1-1} \times \dots \times S_{(p)}^{2n_\ell-1}$ by this p -local homotopy equivalence. Under this assumption, h_i is a unit of $\mathbb{Z}_{(p)}$ for any i . By this decomposition, we can find a projection $\pi_k: G_{(p)} \rightarrow S_{(p)}^{2n_i-1}$ such that $\pi_k^* u_{2n_i-1} = e_i$ for each i . By Lemma 2.3, if $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$, then the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ is nontrivial. As in [KK], if $\langle \epsilon_i, \epsilon_j \rangle$ is nontrivial, then for some $1 \leq k \leq \ell$ we have $n_k + p - 1 = n_i + n_j$ and $\pi_k \circ \langle \epsilon_i, \epsilon_j \rangle$ is nontrivial. Again by Lemma 2.3, this implies that $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$. \square

3. PROOFS OF THE RESULTS

Let p be an odd prime and $p_i, e_n \in H^*(BSO(2n)_{(p)}; \mathbb{Z}/p)$ be the mod p reduction of the i -th universal Pontrjagin class for $i = 1, \dots, n-1$ and the Euler class respectively. Then

$$H^*(BSO(2n)_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[p_1, \dots, p_{n-1}, e_n]$$

and the maps ϵ_i and θ correspond to p_i and e_n respectively in the sense of (2.1). In particular, we take ϵ_i so that $h_i = 1$ for $i \leq \frac{p-1}{2}$ and θ so that $(\Sigma\theta)^* \circ \iota_1^*(e_n) = \Sigma u_{2n-1}$ and $(\Sigma\theta)^* \circ \iota_1^*(p_i) = 0$ for any i .

Lemma 3.1. *The following statements hold.*

- (1) *The element $\mathcal{P}^1 p_i$ does not include the quadratic term $c e_n p_j$ ($c \neq 0$) for any i and j .*
- (2) *If $p = 2n - 1$, the element $\mathcal{P}^1 p_1$ is decomposable and includes the term $(-1)^{\frac{p-1}{2}} e_n^2$.*

Proof. Since $p_i \in H^*(BSO(2n)_{(p)}; \mathbb{Z}/p)$ is contained in the image from $H^*(BSO(2n+1)_{(p)}; \mathbb{Z}/p)$, if a quadratic term of $\mathcal{P}^1 p_i$ includes e_n , it must be a multiple of e_n^2 and $i = n - \frac{p-1}{2} \geq 1$. Thus the first statement holds. Recall that for a maximal torus T of $SO(2n)$ and the natural map $\iota: BT_{(p)} \rightarrow BSO(2n)_{(p)}$, we have

$$H^*(BT_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[t_1, \dots, t_n], \quad |t_i| = 2$$

such that $\iota^*(p_i)$ is the i -th elementary symmetric polynomial in t_1^2, \dots, t_n^2 and $\iota^*(e_n) = t_1 \cdots t_n$. In particular, ι is injective in the mod p cohomology. Suppose $p = 2n - 1$. We have

$$\iota^*(\mathcal{P}^1 p_1) = \mathcal{P}^1(t_1^2 + \cdots + t_n^2) = 2((t_1^2)^{\frac{p+1}{2}} + \cdots + (t_n^2)^{\frac{p+1}{2}}).$$

Then we obtain

$$\mathcal{P}^1 p_1 \equiv (-1)^{\frac{p-1}{2}} e_n^2 \pmod{(p_1, \dots, p_{n-1})^2}$$

by the Newton formula. Therefore the second statement holds. \square

Lemma 3.2. *The element $\mathcal{P}^1 e_n$ is decomposable and the following congruence hold:*

$$\mathcal{P}^1 e_n \equiv (-1)^{\frac{p-1}{2}} \frac{p-1}{2} e_n p_{\frac{p-1}{2}} \pmod{(p_1, \dots, p_{n-1})^2}.$$

Proof. We set $\iota: BT_{(p)} \rightarrow B\mathrm{SO}(2n)_{(p)}$ as in the proof of Lemma 3.1. We have

$$\iota^*(\mathcal{P}^1 e_n) = \mathcal{P}^1 \iota^*(e_n) = \mathcal{P}^1(t_1 \cdots t_n) = t_1 \cdots t_n((t_1^2)^{\frac{p-1}{2}} + \cdots + (t_n^2)^{\frac{p-1}{2}}).$$

Then the proof is completed by the Newton formula. \square

Proof of Theorem 1.3. Assume $p > 2n - 2$. Since the inclusion $\mathrm{SO}(2n - 1)_{(p)} \rightarrow \mathrm{SO}(2n)_{(p)}$ has a left homotopy inverse, it follows from Theorem 1.1 that the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is nontrivial if and only if $2i + 2j > p$. To detect the Samelson products $\langle \epsilon_i, \theta \rangle = \langle \theta, \epsilon_i \rangle$ and $\langle \theta, \theta \rangle$ by Theorem 2.4, we need the information about the quadratic terms of $\mathcal{P}^1 p_i$ and $\mathcal{P}^1 e_n$ including e_n . Now these informations have already been obtained in Lemma 3.1 and 3.2. Therefore the proof of Theorem 1.3 is completed. \square

Lemma 3.2 implies nontriviality of the Samelson product $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle$ not only when $\mathrm{SO}(2n)$ is p -regular but also when $\mathrm{SO}(2n)$ is not p -regular as follows.

Corollary 3.3. *The Samelson product $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle = \langle \theta, \epsilon_{\frac{p-1}{2}} \rangle$ in $\pi_{2n+2p-4}(\mathrm{SO}(2n)_{(p)})$ is nontrivial for any odd prime p . More precisely, the image of $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle$ under the homomorphism induced by the projection $\mathrm{SO}(2n)_{(p)} \rightarrow S_{(p)}^{2n-1}$ generates $\pi_{2n+2p-4}(S_{(p)}^{2n-1}) \cong \mathbb{Z}/p$.*

Proof. Note that, for the projection $\pi: \mathrm{SO}(2n)_{(p)} \rightarrow S_{(p)}^{2n-1}$, we have $(\Sigma\pi)^* \Sigma u_{2n-1} = \iota_1^*(e_n)$. Then the corollary follows from Lemma 2.3 and 3.2. \square

We next prove Theorem 1.4. Let X be a homotopy associative H-space with inverse. For maps $\alpha: A \rightarrow X$ and $\beta: B \rightarrow X$, let $\{\alpha, \beta\}$ denote the composite

$$A \times B \xrightarrow{\alpha \times \beta} X \times X \rightarrow X$$

where the last arrow is the commutator map. Then for the projection $q: A \times B \rightarrow A \wedge B$, we have $q^*(\langle \alpha, \beta \rangle) = \{\alpha, \beta\}$ and the induced map $q^*: [A \wedge B, X] \rightarrow [A \times B, X]$ is injective. In particular, $\langle \alpha, \beta \rangle$ is trivial if and only if $\{\alpha, \beta\}$.

Lemma 3.4 (cf. [KK, Proposition 1]). *For maps $\varphi_i: A_i \rightarrow X$ ($i = 1, 2$) and $\beta: B \rightarrow X$, if $\{\varphi_2, \beta\}$ is trivial, then*

$$\{\varphi_1 \cdot \varphi_2, \beta\} = \{\varphi_1, \beta\} \circ \rho_2$$

where $\rho_i: A_1 \times A_2 \times B \rightarrow A_i$ denotes the projection for $i = 1, 2$.

Proof. In the group of the homotopy set $[A_1 \times A_2 \times B, X]$, we have

$$\{\varphi_1 \cdot \varphi_2, \beta\} = [(\varphi_1 \circ \pi_1) \cdot (\varphi_2 \circ \pi_2), \beta \circ \pi]$$

where $\pi_i: A_1 \times A_2 \times B \rightarrow A_i$ for $i = 1, 2$ and $\pi: A_1 \times A_2 \times B \rightarrow B$ denote the projections and $[-, -]$ means the commutator. In a group G , we have

$$[xy, z] = x[y, z]x^{-1}[x, z]$$

for $x, y, z \in G$. Then the proof is completed by $[\varphi_i \circ \pi_i, \beta \circ \pi] = \{\varphi_i, \beta\} \circ \rho_i$. \square

Proof of Theorem 1.4. Let $\iota: \mathrm{SO}(2n-1) \rightarrow \mathrm{SO}(2n)$ denote the inclusion and $\pi: \mathrm{SO}(2n) \rightarrow S^{2n-1}$ the projection. For $p = 2$ and $n \geq 2$, as remarked in Section 1, the 2-localization $\iota_{(2)}: \mathrm{SO}(2n-1)_{(2)} \rightarrow \mathrm{SO}(2n)_{(2)}$ is not homotopy normal by the argument by James [J, Proof of Theorem (3.1)].

If $2 < p \leq 2n-1$, then the Samelson product

$$\pi_{(p)} \circ \langle \iota_{(p)}, 1_{\mathrm{SO}(2n)_{(p)}} \rangle \circ (\epsilon_{\frac{p-1}{2}} \wedge \theta) = \pi_{(p)} \circ \langle \epsilon_{\frac{p-1}{2}}, \theta \rangle$$

is nontrivial in $\pi_{2n+2p-4}(S_{(p)}^{2n-1})$ by Corollary 3.3. This implies that $\iota_{(p)}$ is not homotopy normal.

Suppose $p > 2n-1$. Note that the identity map of $\mathrm{SO}(2n)_{(p)}$ is identified with the map $\iota_{(p)} \cdot \theta: \mathrm{SO}(2n-1)_{(p)} \times S_{(p)}^{2n-1} \rightarrow \mathrm{SO}(2n)_{(p)}$. Then it follows from Lemma 3.4 that $\iota_{(p)}$ is homotopy normal if the Samelson product $\langle \iota_{(p)}, \theta \rangle$ is trivial. Note also that $\iota_{(p)}$ is identified with the map $\epsilon_1 \cdots \epsilon_{n-1}: S_{(p)}^3 \times \cdots \times S_{(p)}^{4n-5} \rightarrow \mathrm{SO}(2n-1)_{(p)}$. Then it is sufficient to show that $\{\epsilon_1 \cdots \epsilon_{n-1}, \theta\}$ is trivial. By Lemma 3.4, this is equivalent to that $\langle \epsilon_i, \theta \rangle$ are trivial for all i . Thus $\iota_{(p)}$ is homotopy normal by Theorem 1.3. \square

We finally prove Theorem 1.5. Let X, Y be homotopy associative H-spaces with inverse. Recall that the H-deviation $d(f)$ of a map $f: X \rightarrow Y$ is defined by

$$d(f): X \wedge X \rightarrow Y, \quad (x_1, x_2) \mapsto f(x_1 x_2) f(x_2)^{-1} f(x_1)^{-1}.$$

By definition, f is an H-map if and only if the H-deviation $d(f)$ is trivial.

Lemma 3.5. *Let X_1, X_2, Y be homotopy associative H-spaces with inverse, and $\lambda_i: X_i \rightarrow Y$ be H-maps for $i = 1, 2$. Then the map $\lambda_1 \cdot \lambda_2: X_1 \times X_2 \rightarrow Y$ is an H-map if and only if the Samelson product $\langle \lambda_1, \lambda_2 \rangle$ is trivial.*

Proof. For $x_i, x'_i \in X_i$ ($i = 1, 2$), we have

$$\begin{aligned} d(\lambda_1 \cdot \lambda_2)(x_1, x_2, x'_1, x'_2) &\simeq \lambda_1(x_1 x'_1) \lambda_2(x_2 x'_2) \lambda_2(x'_2)^{-1} \lambda_1(x'_1)^{-1} \lambda_2(x_2)^{-1} \lambda_1(x_1)^{-1} \\ &\simeq \lambda_1(x_1) (\langle \lambda_1, \lambda_2 \rangle (x'_1, x_2)) \lambda_1(x_1)^{-1} \end{aligned}$$

since λ_1, λ_2 are H-maps. Then since λ_1 is an H-map, $d(\lambda_1 \cdot \lambda_2)$ is trivial if and only if so is $\langle \lambda_1, \lambda_2 \rangle$, completing the proof. \square

Proof of Theorem 1.5. Obviously, the map $\iota_{(p)} \cdot \theta$ is a homotopy equivalence, so it remains to show that it is an H-map. By definition, we have $d(\theta) \in \pi_{4n-2}(\mathrm{SO}(2n)_{(p)})$, and then by [T, Proposition 13.6] and $p > 2n - 1$, $d(\theta)$ is trivial, implying that θ is an H-map. The inclusion $\iota_{(p)}$ is clearly an H-map, and in the proof of Theorem 1.4 the Samelson product $\langle \iota_{(p)}, \theta \rangle$ is shown to be trivial for $p > 2n - 1$. Thus by Lemma 3.5, $\iota_{(p)} \cdot \theta$ is an H-map. Note that we have not fixed an H-structure of $S_{(p)}^{2n-1}$. There is a one to one correspondence between H-structures on $S_{(p)}^{2n-1}$ and $\pi_{4n-2}(S_{(p)}^{2n-1})$. By [T, Proposition 13.6] and $p > 2n - 1$, $\pi_{4n-2}(S_{(p)}^{2n-1}) = 0$, so there is only one H-structure on $S_{(p)}^{2n-1}$. By [A], $S_{(p)}^{2n-1}$ has a homotopy associative and homotopy commutative H-structure. Then $S_{(p)}^{2n-1}$ must be a homotopy associative and homotopy commutative H-space. \square

REFERENCES

- [A] J.F. Adams, *The sphere, considered as an H-space mod p*, Quart. J. Math. (1961) **12**, 52-60.
- [B] R. Bott, *A note on the Samelson products in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.
- [F] E.M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. of Math. **101** (1975), 510-520.
- [HK1] H. Hamanaka and A. Kono, *A note on the Samelson products in $\pi_*(\mathrm{SO}(2n))$ and the group $[\mathrm{SO}(2n), \mathrm{SO}(2n)]$* , Topology Appl. **154** (2007), no. 3, 567-572.
- [HK2] H. Hamanaka and A. Kono, *A note on Samelson products and mod p cohomology of classifying spaces of the exceptional Lie groups*, Topology Appl. **157** (2010), no. 2, 393-400.
- [HKO] S. Hasui, D. Kishimoto, and A. Ohsita, *Samelson products in p-regular exceptional Lie groups*, Topology Appl. **178** (2014), no. 1, 17-29.
- [J] I.M. James, *On homotopy theory of classical groups*, Ann. Acad. Brasil. Cienc. **39** (1967), 39-44.
- [K] P. G. Kumpel, *Mod p-equivalences of mod p H-spaces*, Quart. J. Math., **23** (1972), 173-178.
- [KK] S. Kaji and D. Kishimoto, *Homotopy nilpotency in p-regular loop spaces*, Math. Z., **264** (2010), no.1, 209-224.
- [KO] A. Kono, H. Ōshima, *Commutativity of the group of self homotopy classes of Lie groups*, Bull. London Math. Soc. **36** (2004) 37-52.
- [L] J. Lin, *H-spaces with Finiteness Conditions*, Handbook of Algebraic Topology, North-Holland Elsevier (1995), pp. 1095-1141 Chapter 22.
- [Ma] M. Mahowald, *A Samelson product in $\mathrm{SO}(2n)$* , Bol. Soc. Math. Mexicana **10** (1965) 80-83.
- [Mo] K. Morisugi, *Hopf construction, Samelson products and suspension maps*, Contemporary Math. **239** (1999), 225-238.
- [T] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.

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